

# Formality of Cyclic Chains

**Thomas Willwacher**

Department of Mathematics, ETH Zurich, Zurich, Switzerland

*Correspondence to be sent to: thomas.willwacher@gmail.com*

We prove the cyclic formality conjecture for chains, raised by Tsygan ["Formality conjectures for chains." *Differential Topology, Infinite-dimensional Lie Algebras, and Applications* 261–74. American Mathematical Society Translation Series 2, 194. Providence, RI: American Mathematical Society, 1999.]. It states the existence of an  $L_\infty$ -quasi-isomorphism of  $L_\infty$ -modules between the cyclic chain complex of smooth functions on a manifold and the differential forms on that manifold. Concretely, we prove that the  $u$ -linear extension of Shoikhet's morphism of Hochschild chains [Shoikhet, B. "A proof of the Tsygan formality conjecture for chains." *Advances in Mathematics* 179, no. 1 (2003): 7–37.] solves Tsygan's conjecture.

## 1 Introduction and Notation

Let  $M$  be a smooth manifold and  $T_{\text{poly}}^\bullet = \Gamma(M; \bigwedge^\bullet TM)$  be the space of polyvector fields on  $M$ . The Schouten–Nijenhuis bracket  $[\cdot, \cdot]_{\text{SN}}$  endows  $T_{\text{poly}}^{\bullet+1}$  with the structure of a graded Lie algebra. Denote by  $A = C^\infty(M)$ , the commutative algebra of smooth functions on  $M$ . Let  $D_{\text{poly}}^\bullet$  be the subcomplex of the Hochschild complex  $C^\bullet(A, A)$  given by polydifferential operators. The  $n$ -cochains in this complex are spanned by maps of the form

$$A^{\otimes n} \ni a_1 \otimes \cdots \otimes a_n \mapsto \prod_{k=1}^n (D_k a_k) \in A,$$

Received March 9, 2010; Revised September 5, 2010; Accepted September 7, 2010  
 Communicated by Prof. Anton Alekseev

where the  $D_k$  are differential operators. The Hochschild differential  $d_H$  and the Gerstenhaber bracket  $[\cdot, \cdot]_G$  naturally restrict to this subcomplex and endow  $D_{\text{poly}}^{\bullet+1}$  with the structure of a differential graded Lie algebra (dgla).

In his famous paper, Kontsevich [7] proved in 1997 the *Formality Theorem* (on cochains), that is, the existence of an  $L_\infty$ -quasi-isomorphism of dgla's

$$T_{\text{poly}}^{\bullet+1} \rightarrow D_{\text{poly}}^{\bullet+1}.$$

The Taylor coefficients of this morphism were explicitly given in terms of graphs. Kontsevich's techniques for dealing with graphs and constructing proofs based on Stokes' Theorem are very relevant for most papers on the subject, and this paper is no exception. However, we will not review the theory here, but refer the reader to the original work [7].

Next, consider the completed Hochschild chain complex  $C_\bullet(A, A)$  of  $A$  with values in  $A$ . (See Remark A.2 in the appendix) It forms a dgla module over  $D_{\text{poly}}^{\bullet+1}(A, A)$ , with the action given by

$$\begin{aligned} D \cdot (a_0 \otimes \cdots \otimes a_n) &= \sum_{j=n-d+1}^n (-1)^{n(j+1)} D(a_{j+1}, \dots, a_0, \dots) \otimes a_{d+j-n} \otimes \cdots \otimes a_j \\ &+ \sum_{i=0}^{n-d} (-1)^{(d-1)(i+1)} a_0 \otimes \cdots \otimes a_i \otimes D(a_{i+1}, \dots, a_{i+d}) \otimes \cdots \otimes a_n \end{aligned} \quad (1)$$

for  $D \in D_{\text{poly}}^d(A, A)$  and  $a_0, \dots, a_n \in A$ . Through Kontsevich's morphism the chains  $C_\bullet(A, A)$  also carry an  $L_\infty$ -module structure over the dgla  $T_{\text{poly}}^{\bullet+1}$ .

Furthermore, there is another natural module over  $T_{\text{poly}}^{\bullet+1}$  that can be constructed without additional data, namely the differential forms  $\Omega^\bullet(M)$ , with the action given by Lie derivatives

$$\gamma \cdot \omega = (d_\gamma - (-1)^p \iota_\gamma d) \omega,$$

where  $\gamma \in T_{\text{poly}}^p$  and  $\omega \in \Omega^\bullet(M)$ . Here  $\iota_\gamma \omega$  denotes the contraction of  $\gamma$  with  $\omega$  as usual. (Concretely, it is defined such that  $\iota_{\gamma \wedge v} = \iota_\gamma \iota_v$  for  $\gamma, v \in T_{\text{poly}}^\bullet$ .) A natural extension of the formality Theorem is then the following statement, which was conjectured by Tsygan [11] in 1999.

**Theorem 1.1** (Formality Theorem on Chains [3, 8, 10]). There exists an  $L_\infty$ -quasi-isomorphism of  $L_\infty$  modules over  $T_{\text{poly}}^{\bullet+1}$

$$\mathcal{U}: (C_\bullet(A, A), b) \rightarrow (\Omega^\bullet(M), 0). \quad \square$$

Here the notation means that the complex  $\Omega^\bullet(M)$  is endowed with 0 differential. The Theorem has been proved by Shoikhet [8] and Dolgushev [3] and independently by Tamarkin and Tsygan [10]. More precisely, Shoikhet found an explicit quasi-isomorphism  $\mathcal{U}^{\text{sh}}$  in the cases  $M = \mathbb{R}^n$ , or  $M$  a formal completion of  $\mathbb{R}^n$  at the origin. Dolgushev globalized this construction using Fedosov resolutions. The explicit construction of  $\mathcal{U}^{\text{sh}}$  given by Shoikhet will be reviewed in Section 2. The proof given by Tamarkin and Tsygan in [10] of the above Theorem is independent of that of Shoikhet and Dolgushev. It is essentially a “modules”-version of Tamarkin’s nonconstructive proof of Kontsevich’s Formality Theorem [6, 9]. It will play no role in this paper.

Tsygan also conjectured the analog of the above theorem on cyclic instead of Hochschild chains. This is the conjecture that will be proved in this paper. There are several variants of the cyclic chain complex, all of which have the form

$$CC_p^W(A) = \left( C_\bullet(A, A)[[u]] \bigotimes_{\mathbb{C}[u]} W \right)_p, \quad (2)$$

where  $W$  is a module over the graded algebra  $\mathbb{C}[u]$ , with  $u$  being a formal variable of degree  $+2$ . (This notation is due to Getzler. Note that we use the negative grading on chains.) The differential on the above complexes is given by  $b + uB$ , where  $b$  is the Hochschild boundary operator and  $B$  is defined by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^n (-1)^{nj} 1 \otimes a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}, \quad (3)$$

where  $a_{-1} := a_n$  to simplify notation. The homology  $HC_\bullet^W(A)$  of the cyclic chain complex is related to the de Rham cohomology of  $M$  via the following theorem, which can be found in [1] (Theorem 3.3 for  $G = \{pt\}$ ).

**Theorem 1.2.** Let  $W$  be a  $\mathbb{C}[u]$ -module of finite projective dimension over  $\mathbb{C}[u]$ , then

$$HC_{\bullet}^W(A) \cong H^{\bullet} \left( \Omega(M)[[u]] \bigotimes_{\mathbb{C}[u]} W, ud \right). \quad \square$$

We will prove the following theorem.

**Theorem 1.3.** Shoikhet's  $L_{\infty}$ -morphism  $\mathcal{U}^{\text{sh}}$  satisfies

$$\mathcal{U}^{\text{sh}} \circ B = d \circ \mathcal{U}^{\text{sh}}. \quad \square$$

As a corollary, one obtains the formality theorem on cyclic chains.

**Corollary 1.4.** For  $W$  a  $\mathbb{C}[u]$ -module of finite projective dimension over  $\mathbb{C}[u]$ , there is an  $L_{\infty}$ -quasi-isomorphism of  $L_{\infty}$ -modules over  $T_{\text{poly}}$

$$\mathcal{U} : (CC_{\bullet}^W(A, A), b + uB) \rightarrow \left( \Omega(M)[[u]] \bigotimes_{\mathbb{C}[u]} W, ud \right). \quad \square$$

**Proof.** For the proof, one needs to consider Fedosov resolutions of the above two complexes. Introducing these and the required notation would be very lengthy. To avoid this, we take the liberty to copy the notation of Dolgushev, as used in [3, Section 5], until the end of this proof. For definitions and explanations, we refer to Dolgushev's diligent treatment. Concretely, there is the following sequence of quasi-isomorphisms of  $L_{\infty}$ -modules over  $T_{\text{poly}}$ :

$$C^{\text{poly}}(M) \xrightarrow{\rho} (\Omega(M, C^{\text{poly}}), D + \flat) \xrightarrow{\mathfrak{R}} (\Omega(M, \mathcal{E}), D) \xleftarrow{\tau} \mathcal{A}^{\bullet}(M).$$

From left to right, the objects are the Hochschild chain complex of  $C^{\infty}(M)$ , its Fedosov resolution, the Fedosov resolution of the de Rham complex, and the de Rham complex itself. The middle quasi-isomorphism (i.e.,  $\mathfrak{R}$ ) is defined using Shoikhet's morphism  $\mathcal{U}^{\text{sh}}$  fiberwise.

All the above four complexes are, in fact, *mixed complexes*, in the sense that they carry another differential of degree  $-1$ , anticommuting with their boundary operators. This differential is (from left to right) Connes'  $B$  as in (3), the same operator applied fiberwise  $B_f$ , the fiberwise de Rham differential  $d_f$ , and finally the de Rham differential  $d$ . We claim that all morphisms in the above sequence are morphisms of mixed complexes, that is, commute with the application of the additional differentials. For the

middle morphism  $\mathfrak{R}$ , this follows from Theorem 1.3. For the left- and rightmost morphisms, note that the fiberwise  $B_f$  and  $d_f$  map  $D$ -constant sections to  $D$ -constant sections. Hence it suffices to observe that for  $s \in C^{\text{poly}}(M)$  and  $\alpha \in \mathcal{A}^\bullet(M)$ , the parts of degree 0 in the formal variable (usually called “ $y$ ”) of  $B_f \rho(s)$  and  $d_f \tau(\alpha)$  agree with  $Bs$  and  $d\alpha$ , respectively.

By  $u$ -linear extension and Remark A.1 in the appendix, we then obtain the following sequence of morphisms of  $L_\infty$ -modules over  $T_{\text{poly}}$ :

$$\begin{aligned} \left( C^{\text{poly}}(M)[[u]] \bigotimes_{\mathbb{C}[u]} W, \mathfrak{b} + uB \right) &\longrightarrow \left( \Omega(M, C^{\text{poly}})[[u]] \bigotimes_{\mathbb{C}[u]} W, D + \mathfrak{b} + uB_f \right) \\ &\longrightarrow \left( \Omega(M, \mathcal{E})[[u]] \bigotimes_{\mathbb{C}[u]} W, D + ud_f \right) \longleftarrow \left( \mathcal{A}^\bullet(M)[[u]] \bigotimes_{\mathbb{C}[u]} W, ud \right). \end{aligned}$$

It remains to be shown that all these morphisms are quasi-isomorphisms. For this, one can forget about the higher degree Taylor components of the  $L_\infty$ -module-morphisms and consider the above sequence as a sequence of morphisms of complexes. But we know that the (0th Taylor components of the) original morphisms  $\rho$ ,  $\mathfrak{R}$ , and  $\tau$  were morphisms of mixed complexes inducing isomorphisms on homology (w.r.t. the degree  $-1$  differential). Hence [5, Proposition 2.4] finishes the proof of the Theorem. ■

We want to mention that Corollary 1.4 also follows from the more general  $\text{Calc}_\infty$  Formality Theorem of Dolgushev et al. [4], which appeared after the writing of this manuscript.

### 1.1 Structure of the paper

The precise definitions of structures, brackets, differentials, and gradings that were omitted in the introduction can be found in the appendix. The author wishes to avoid having the reader browse through pages of definitions she or he already knows. So in the next section, we directly start by reviewing the construction of Shoikhet’s formality morphism, adding several remarks that will simplify the proof of Theorem 1.3. The proof can then be found in Section 3.

## 2 Shoikhet's Formality Theorem on Chains

In this section, we recall the construction of Shoikhet's morphism  $\mathcal{U}^{\text{sh}}$  for the case  $M = \mathbb{R}^d$  and outline his proof of Theorem 1.1. As usual in deformation quantization, the morphism can be expressed as a sum of graphs. Denote by  $\mathcal{U}_m^{\text{sh}}$  the  $m$ th Taylor component of  $\mathcal{U}^{\text{sh}}$ . For  $\xi$  a constant polyvector field, we will set

$$\left( \iota_{\xi} \mathcal{U}_m^{\text{sh}}(\gamma_1, \dots, \gamma_m; a_0 \otimes \dots \otimes a_n) \right)_0 = \pm \sum_{\Gamma \in G(m, n)} w_{\Gamma} D_{\Gamma}(\xi, \gamma_1, \dots, \gamma_m; a_0 \otimes \dots \otimes a_n).$$

The sign on the right-hand side depends on conventions. In our conventions, it is  $\pm = (-1)^{|\gamma_1| + \dots + |\gamma_m|}$  for homogeneous  $\gamma_1, \dots, \gamma_m$ . In this paper, however, the sign will not play any role. On the left, the notation  $(\dots)_0$  means that one picks out the 0-form part. On the right, the sum is over all Kontsevich graphs with  $m+1$  type I and  $n+1$  type II vertices. The polydifferential operator  $D_{\Gamma}$  is the same as in the Kontsevich case, but with the polyvector field  $\xi$  put exclusively at the first vertex of the graph, cf. [8, Section 2.2.4]. However, the weight  $w_{\Gamma} \in \mathbb{R}$  is defined differently; a formula will be given below. In particular, the weight is defined in such a way that the right-hand side of the above equation vanishes if the polyvector field  $\xi$  does not have the appropriate degree.

To be precise, we will use here the following definition of the graphs occurring in the sum.

**Definition 2.1.** The set  $G(m, n)$ ,  $m, n \in \mathbb{N}_0$  consists of directed graphs  $\Gamma$  such that

- The vertex set of  $\Gamma$  is

$$V(\Gamma) = \{0, 1, \dots, m\} \cup \{\bar{0}, \dots, \bar{n}\},$$

where the vertex 0 will be called the *central* vertex, the vertices  $\{0, 1, \dots, m\}$  the *type I* vertices and the  $\{\bar{0}, \dots, \bar{n}\}$  the *type II* vertices.

- Every edge  $e = (v_i \rightarrow v_j) \in E(\Gamma)$  starts at a type I vertex and does not end at the central vertex. That is,  $v_i$  is type I and  $v_j$  is not the vertex 0. We will call the edges  $(0 \rightarrow v_k)$  that start at the central vertex *central edges* and denote the set of these edges by  $E_c(\Gamma)$ .
- There are no tadpoles, that is, no edges of the form  $(v \rightarrow v)$ .

- For each type I vertex  $v$ , there is an ordering given on

$$\text{Star}(v) = \{(v \rightarrow w) \mid (v \rightarrow w) \in E(\Gamma), w \in E(\Gamma)\}.$$

This ordering is considered part of the data.  $\square$

Let us next define the weight  $w_\Gamma$  of  $\Gamma \in G(m, n)$ . As in the Kontsevich case, it is an integral of a certain differential form over a compact manifold with corners, the *configuration space*  $C_\Gamma$ .

$$w_\Gamma = \left( \prod_{v \in V(\Gamma)} \frac{1}{(\#\text{Star}(v))!} \right) \int_{C_\Gamma} \omega_\Gamma. \quad (4)$$

**Definition 2.2.** The configuration space  $C_\Gamma$  is the Fulton–MacPherson-like compactification of the space of embeddings

$$(z_0, \dots, z_m, z_{\bar{0}}, \dots, z_{\bar{n}}) : V(\Gamma) \longrightarrow D$$

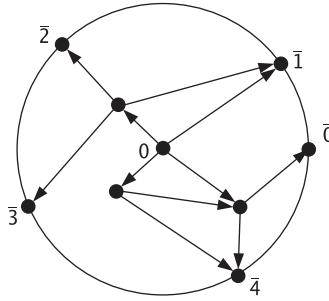
of the vertex set  $V(\Gamma)$  of  $\Gamma$  into the closed unit disk  $D = \{z \in \mathbb{C}; |z| \leq 1\}$  such that

- (1) The central vertex is mapped to the origin, that is,  $z_0 = 0$ .
- (2) The vertex  $\bar{0}$  is mapped to 1, that is,  $z_{\bar{0}} = 1$ .
- (3) All type I vertices are mapped to the interior of  $D$ , that is,  $z_j \in D^\circ$  for  $j = 1, \dots, n$ .
- (4) All type II vertices are mapped to the boundary of  $D$ , that is,  $z_{\bar{j}} \in \partial D$  for  $j = 0, \dots, m$ .
- (5) The type II vertices occur in counterclockwise increasing order on the circle, that is,  $0 < \arg \frac{z_{\bar{1}}}{z_0} < \dots < \arg \frac{z_{\bar{n}}}{z_0} < 2\pi$ . (We mean the compactification constructed similarly to [7, Section 5]. It will not be of any importance.)  $\square$

An example graph embedded in  $D$  is shown in Figure 1.

The differential form  $\omega_\Gamma$  that is integrated over the configuration space can be expressed as a product of one-forms, one for each edge in  $\Gamma$ .

$$\omega_\Gamma = \bigwedge_{(0 \rightarrow K) \in E_c(\Gamma)} d\theta_c(z_K, z_0) \wedge \bigwedge_{j=1}^m \bigwedge_{(j \rightarrow L) \in E(\Gamma)} d\theta(z_j, z_L). \quad (5)$$



**Fig. 1.** Some Shoikhet graph. The set of central edges  $E_c(\Gamma)$  in this case consists of the four edges starting at the vertex “0” in the middle.

Here the one-forms occurring are defined as

$$d\theta_c(z, w) = -\frac{1}{2\pi} d\arg\left(\frac{z}{w}\right), \quad (6)$$

$$d\theta(z, w) = \frac{1}{2\pi} d\arg((z - w)(1 - z\bar{w})\bar{z}). \quad (7)$$

The geometric meaning of these forms is illustrated in Figure 2. The ordering of the forms within the wedge products is such that forms corresponding to edges with source vertex  $j$  stand on the left of those with source vertex  $j + 1$ , and according to the order given on the stars for edges having the same source vertex.

We will use the abbreviations

$$\omega_\Gamma^c = \bigwedge_{(0 \rightarrow K) \in E_c(\Gamma)} d\theta_c(z_K, z_0) \omega_\Gamma^{\text{nc}} = \bigwedge_{j=1}^m \bigwedge_{(j \rightarrow L) \in E(\Gamma)} d\theta(z_j, z_L)$$

for the factors of  $\omega_\Gamma = \omega_\Gamma^c \wedge \omega_\Gamma^{\text{nc}}$  coming from central and noncentral edges.

**Remark 1.** Note that the form  $d\theta_c(z, w)$  satisfies  $d\theta_c(z, w) = d\theta_c(z, u) + d\theta_c(u, w)$  for any  $u \in D \setminus \{0\}$ .  $\square$

**Remark 2.** On  $C_\Gamma$  we put the orientation induced by the volume form

$$i^m dz_1 d\bar{z}_1 \dots dz_m d\bar{z}_m d\arg z_{\bar{n}} \dots d\arg z_{\bar{1}}.$$

$\square$



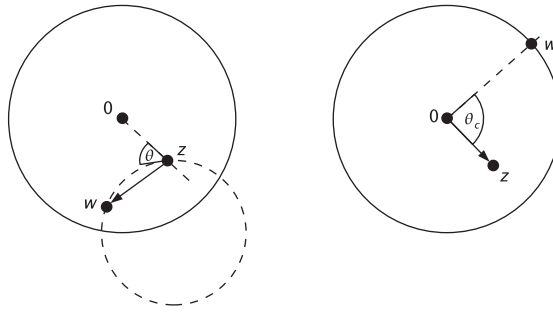


Fig. 2. Geometric meaning of Shoikhet's angle forms.

### 3 Proof of Theorem 1.3

We have to show that

$$\begin{aligned} & \left( \iota_{\xi} \left( d\mathcal{U}_m^{\text{sh}}(\gamma_1, \dots, \gamma_m)(a_0 \otimes \dots \otimes a_n) \right) \right)_0 \\ &= (-1)^{|\gamma_1| + \dots + |\gamma_m|} \left( \iota_{\xi} \mathcal{U}_m^{\text{sh}}(\gamma_1, \dots, \gamma_m)(B(a_0 \otimes \dots \otimes a_n)) \right)_0 \end{aligned} \quad (8)$$

for any polyvector field  $\xi$  of degree  $p = n + 2m - |\gamma_1| - \dots - |\gamma_m| + 1$ . In fact, we will show that both sides of the above equation equal the following expression.

$$(-1)^{p-1} \sum_{\Gamma \in G(m, n)} w_{\Gamma - [e]} D_{\Gamma}(\xi, \gamma_1 \wedge \dots \wedge \gamma_m; a_0 \otimes \dots \otimes a_n). \quad (9)$$

Here  $e$  is the first edge in  $E_c(\Gamma) = \text{Star}(0)$ .

**Lemma 3.1.** The left-hand side of (8) is equal to (9). □

**Proof.** We can assume w.l.o.g. that  $\xi = \xi_1 \wedge \dots \wedge \xi_p$ , with the  $\xi_j$  constant vector fields. Then, for any  $p$ -form  $\omega$ , we have

$$\iota_{\xi} d\omega = \iota_{\xi_1} \dots \iota_{\xi_p} d\omega = (-1)^{p-1} \sum_{i=1}^p (-1)^{i+1} L_{\xi_i} \iota_{\xi_1} \dots \hat{\iota}_{\xi_i} \dots \iota_{\xi_p} \omega.$$

Here  $L_{\xi_i}$  denotes the derivative in the direction  $\xi_i$ . On the other hand, we have

$$\begin{aligned} & \sum_{i=1}^p (-1)^{i+1} L_{\xi_i} D_{\Gamma}(\xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p, \gamma_1, \dots, \gamma_m; a_0 \otimes \cdots \otimes a_n) \\ &= \sum_{v \in V(\Gamma) \setminus \{0\}} D_{\Gamma \cup \{(0 \rightarrow v)\}}(\xi, \gamma_1, \dots, \gamma_m, a_0 \otimes \cdots \otimes a_n). \end{aligned}$$

Here by  $\Gamma \cup \{(0 \rightarrow v)\}$  we mean the graph formed by adding the edge  $(0 \rightarrow v)$  to  $\Gamma$  and adjusting the ordering in  $E_c(\Gamma)$ , so that the newly added edge is the first. Next multiply by  $w_{\Gamma}$  and sum over all graphs  $\Gamma$ . Observe that the double sum occurring, namely

$$\sum_{\Gamma \in G(m, n)} \sum_{v \in V(\Gamma) \setminus \{0\}}$$

contains every graph in  $G(m, n)$  with at least one central edge exactly once. (Recall that the data of a graph contains an ordering on the stars.) The Lemma hence follows by changing summation variables.  $\blacksquare$

**Lemma 3.2.** The right-hand side of (8) is equal to (9).  $\square$

For the proof, we need some preparation. First define the operator  $\sigma$  (cyclic shift) on  $C_{\bullet}(A, A)$  by

$$\sigma(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes a_0.$$

Also define the operator  $s$  on  $C_{\bullet}(A, A)$  by

$$s(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

so that  $B = \sum_{i=0}^n (-1)^{in} s \sigma^i$ . Similarly, one can define operators  $s$  and  $\sigma$  on the space of graphs such that

$$\begin{aligned} & \sum_{\Gamma \in G(m, n+1)} w_{\Gamma} D_{\Gamma}(\xi, \gamma_1, \dots, \gamma_m; B(a_0 \otimes \cdots \otimes a_n)) \\ &= \sum_{\Gamma \in G(m, n)} \left( \sum_{i=0}^n w_{s \sigma^i \Gamma} \right) D_{\Gamma}(\xi, \gamma_1, \dots, \gamma_m; a_0 \otimes \cdots \otimes a_n). \end{aligned}$$

Concretely, the operator  $\sigma$  on graphs performs a cyclic relabeling of the type II vertices. The operator  $s$  adds a new type II vertex, which gets labeled  $\bar{0}$ . The above lemma is then an easy consequence of the following result.

**Lemma 3.3.** Let  $\Gamma$  be a graph with  $n$  type II vertices. Then

$$\sum_{i=0}^n (-1)^{in} w_{s\sigma^i \Gamma} = \frac{(-1)^n}{\#E_c(\Gamma)} \sum_{i=1}^{\#E_c(\Gamma)} (-1)^{i+1} w_{\Gamma - \{e_i\}}.$$

where  $e_i$  is the  $i$ th edge in  $E_c(\Gamma)$ . □

**Proof.** Consider the family of maps  $\rho_i: C_{s\sigma^i \Gamma} \rightarrow S^1 \times C_\Gamma$  defined such that

$$(z_1, \dots, z_m, z_{\bar{1}}, \dots, z_{\overline{n+1}}) \mapsto (z_{i+1}^{-1}, (z_{i+1}^{-1} z_1, \dots, z_{i+1}^{-1} z_m, z_{i+1}^{-1} z_{i+2}, \dots, z_{i+1}^{-1} z_{n+1}, z_{i+1}^{-1} z_{\bar{1}}, \dots)).$$

If one puts the natural orientation on  $S^1 \times C_\Gamma$ , then one can check that  $\rho_i$  changes orientation by a factor  $(-1)^{n+\bar{n}+1}$ . Furthermore, the maps  $\rho_i$  are injections and local diffeomorphisms on the interior, and hence one can write

$$w_{s\sigma^i \Gamma} = \left( \prod_{v \in V(\Gamma)} \frac{1}{(\#\text{Star}(v))!} \right) \int_{\text{Im}(\rho_i)} (-1)^{\bar{n}+n+1} (\rho_i)_* \omega_{s\sigma^i \Gamma}.$$

In the following, we will use a coordinate  $Z$  on  $S^1 \subset \mathbb{C}$  and denote by  $z_0, \dots, z_m, z_{\bar{0}}, \dots, z_{\bar{n}}$  the standard coordinates on  $C_\Gamma$ . Note that the integrand can be written as follows (cf. Equations (5)–(7)):

$$(\rho_i)_* \omega_{s\sigma^i \Gamma} = \bigwedge_{(0 \rightarrow K) \in E_c(\Gamma)} (d\theta_c(z_K, z_{\bar{0}}) + d\theta_c(z_{\bar{0}}, Z)) \wedge \omega_{\Gamma}^{\text{nc}}.$$

Here we used Remark 1. Note also that  $z_{\bar{0}} = 1$  by our convention, and that the expression on the right is independent of  $i$ . The images  $\text{Im}(\rho_i)$  of the maps  $\rho_i$ ,  $i = 0, \dots, n$ , cover the

space  $S^1 \times C_\Gamma$  and their intersections are of higher codimension. One can hence write

$$\begin{aligned} \sum_{i=0}^n (-1)^{in} w_{s\sigma^i \Gamma} &= \sum_{i=0}^n (-1)^{in} \left( \prod_{v \in V(\Gamma)} \frac{1}{(\#\text{Star}(v))!} \right) \int_{\text{Im}(\rho_i)} (-1)^{ni+n+1} (\rho_i)_* \omega_{s\sigma^i \Gamma} \\ &= -(-1)^n \left( \prod_{v \in V(\Gamma)} \frac{1}{(\#\text{Star}(v))!} \right) \int_{S^1 \times C_\Gamma} \bigwedge_{(0 \rightarrow K) \in E_c(\Gamma)} \\ &\quad \times (\text{d}\theta_c(z_K, z_{\tilde{0}}) + \text{d}\theta_c(z_{\tilde{0}}, Z)) \wedge \omega_\Gamma^{\text{nc}}. \end{aligned}$$

We can then perform the integration over the  $S^1$ -factor and obtain

$$\begin{aligned} \sum_{i=0}^n (-1)^{in} w_{s\sigma^i \Gamma} &= (-1)^n \left( \prod_{v \in V(\Gamma)} \frac{1}{(\#\text{Star}(v))!} \right) \sum_{i=1}^{\#E_c(\Gamma)} (-1)^{i+1} \int_{C_\Gamma} \bigwedge_{(0 \rightarrow K) \in E_c(\Gamma - \{e_i\})} \text{d}\theta_c(z_K, z_{\tilde{0}}) \wedge \omega_\Gamma^{\text{nc}} \\ &= \frac{(-1)^n}{\#E_c(\Gamma)} \sum_{i=1}^{\#E_c(\Gamma)} (-1)^{i+1} w_{\Gamma - \{e_i\}}. \quad \blacksquare \end{aligned}$$

**Proof of Lemma 3.2.** The remainder of the proof of Lemma 3.2, and hence of Theorem 1.3 is just a straightforward check of the signs, using that

$$p-1 \equiv n + |\gamma_1| + \cdots + |\gamma_m| \pmod{2}. \quad \blacksquare$$

### Acknowledgements

The author is very grateful to his advisor Prof. Giovanni Felder for introducing him to the problem and many helpful discussions and corrections to this manuscript.

### Funding

This work has been partially supported by the Swiss National Science Foundation, grant 200020-105450.

### Appendix. Standard Definitions

In this section, we state some standard definitions and results. We mostly use the terminology of Tsygan [11], and hence almost copy the exposition given in his paper.

A.1  $L_\infty$ -algebras and  $L_\infty$ -modules

Let  $\mathfrak{g}^\bullet$  be a  $\mathbb{Z}$ -graded vector space. An  $L_\infty$ -structure on  $\mathfrak{g}^\bullet$  is a degree 1 coderivation  $Q$  on the cocommutative cofree coalgebra without counit  $S(\mathfrak{g}^{\bullet+1})$  satisfying

$$Q^2 = 0.$$

Any coderivation on  $S(\mathfrak{g}^{\bullet+1})$  is determined by its projection to  $\mathfrak{g}^\bullet$ , hence by a series of linear functions

$$q_k \in \text{Hom} \left( \bigwedge^k \mathfrak{g}^\bullet, \mathfrak{g}^\bullet \right)$$

of degree  $2 - k$ . The condition that  $Q^2 = 0$  reads

$$\sum_{j=1}^N \sum_{\sigma \in S_N} \pm \frac{1}{j!(N-j)!} q_{N-j+1}(q_j(a_{\sigma(1)}, \dots, a_{\sigma(j)}), a_{\sigma(j+1)}, \dots, a_{\sigma(N)}) = 0$$

for all  $N = 1, 2, \dots$  and all  $a_1, \dots, a_N \in \mathfrak{g}^\bullet$ . Here the sign is the lexicographic sign w.r.t. the shifted-by-one grading.

Let now  $M^\bullet$  be another graded vector space. An  $L_\infty$ -module structure on  $M^\bullet$  is a degree 1 coderivation  $D$  on the cofree comodule

$$S(\mathfrak{g}^{\bullet+1}) \otimes M^\bullet$$

satisfying  $D^2 = 0$ . Again,  $D$  is determined by its composition with the projection to  $M^\bullet$ , that is, by components

$$d_k \in \text{Hom} \left( \bigwedge^k \mathfrak{g}^\bullet \otimes M^\bullet, M^\bullet \right)$$

of degree  $1 - k$  such that the following holds for all  $N = 1, 2, \dots$  and  $a_1, \dots, a_N \in \mathfrak{g}^\bullet$ ,  $m \in M^\bullet$ :

$$\begin{aligned} \sum_{j=1}^N \sum_{\sigma \in S_N} \left[ \pm \frac{1}{j!(N-j)!} d_{N-j}(a_{\sigma(1)}, \dots, a_{\sigma(j)}, d_j(a_{\sigma(j+1)}, \dots, a_{\sigma(N)}, m)) \right. \\ \left. \pm \frac{1}{j!(N-j)!} d_{N-j+1}(q_j(a_{\sigma(1)}, \dots, a_{\sigma(j)}), a_{\sigma(j+1)}, \dots, a_{\sigma(N)}, m) \right] = 0. \end{aligned}$$

Morphisms of  $L_\infty$ -algebras and  $L_\infty$ -modules are defined in the obvious way as morphisms of the underlying coalgebras or comodules that commute with the structure ( $Q$  or  $D$ ) given.

Philosophically, and also mathematically if  $\dim \mathfrak{g}^\bullet < \infty$ , then one can understand the components  $q_k$  of  $Q$  as terms in a “Taylor series”

$$Q = \sum_{k \geq 1} \frac{q_k}{k!}$$

of a degree 1 vector field  $Q$  on  $\mathfrak{g}^{\bullet+1}$ , commuting with itself. Consider next the trivial bundle  $\mathfrak{g}^{\bullet+1} \otimes M^\bullet \rightarrow \mathfrak{g}^{\bullet+1}$ . An  $L_\infty$ -module structure can be understood philosophically as a flat lift  $D$  of the vector field  $Q$  to this bundle.

**Remark A.1.** The only way in which the above definitions are needed in this paper is the following. Consider an  $L_\infty$ -algebra  $(\mathfrak{g}^\bullet, Q)$  as above and a morphism  $\mathcal{U}$  of  $L_\infty$ -modules over  $\mathfrak{g}^\bullet$

$$\mathcal{U} : (M_1^\bullet, D_1) \longrightarrow (M_2^\bullet, D_2).$$

We next wish to modify the  $L_\infty$ -module structures to

$$D'_1 = D_1 + \delta_1,$$

$$D'_2 = D_2 + \delta_2,$$

where the  $\delta_j$  are degree 1 endomorphisms of  $S\mathfrak{g}^{\bullet+1} \otimes M_j^\bullet$ . Then  $\mathcal{U}$  is still a morphism of the new  $L_\infty$ -modules  $(M_j^\bullet, D'_j)$  if and only if

$$\mathcal{U} \circ \delta_1 = \delta_2 \circ \mathcal{U}.$$

As usual, it is sufficient to consider the projection of both sides to  $M_2^\bullet$ , because  $D'_j$  are coderivations. In our case furthermore, all Taylor components of the  $\delta_j$  vanish except in degree 0. Hence the above condition reads in components

$$\mathcal{U}_N(a_1, \dots, a_N, \delta_1 m) = (-1)^{|a_1| + \dots + |a_N|} \delta_2 \mathcal{U}_N(a_1, \dots, a_N, m)$$

for  $N = 0, 1, \dots$ . This is precisely the condition (8) proved in Section 3. Here the degrees are the degrees in the coalgebra, that is,  $a_j \in \mathfrak{g}^{|a_j|+1}$ .  $\square$

## A.2 Polyvector fields

The Schouten–Nijenhuis bracket  $[\cdot, \cdot]_{\text{SN}}$  on  $T_{\text{poly}}^\bullet$  is defined such that

$$[f, g]_{\text{SN}} = 0,$$

$$[\xi, \gamma_1]_{\text{SN}} = L_\xi \gamma_1,$$

$$[\gamma_1, \gamma_2 \wedge \gamma_3]_{\text{SN}} = [\gamma_1, \gamma_2]_{\text{SN}} \wedge \gamma_3 + (-1)^{(|\gamma_1|-1)|\gamma_2|} \gamma_2 \wedge [\gamma_1, \gamma_3]_{\text{SN}}$$

for all functions  $f \in A$ , vector fields  $\xi \in T_{\text{poly}}^0$  and polyvector fields  $\gamma_1, \gamma_2, \gamma_3 \in T_{\text{poly}}^\bullet$ . One can check that the above bracket turns  $T_{\text{poly}}^{\bullet+1}$  into a graded Lie algebra. As any Lie algebra, it is automatically an  $L_\infty$ -algebra, obtained by setting

$$q_k(\gamma_1, \dots, \gamma_k) = \begin{cases} (-1)^{|\gamma_1|} [\gamma_1, \gamma_2]_{\text{SN}} & \text{for } k=2, \\ 0 & \text{otherwise.} \end{cases}$$

Next consider the space  $\Omega^\bullet(M)$  of differential forms on the manifold  $M$ . We consider it with the opposite of the usual grading, that is, a  $k$ -form has degree  $-k$ . With this grading,  $\Omega^\bullet(M)$  is a graded module over the graded Lie algebra  $T_{\text{poly}}^\bullet$ . The action is given by

$$L_\gamma \omega = [d, \iota_\gamma] \omega$$

for polyvector fields  $\gamma$  and differential forms  $\omega$ . For a function  $f \in T_{\text{poly}}^0$  we define  $\iota_f$  to be the multiplication by  $f$ . Any module over a Lie algebra is also an  $L_\infty$ -module, in this case by setting

$$d_k(\gamma_1, \dots, \gamma_k, \omega) = \begin{cases} (-1)^{|\gamma_1|} L_{\gamma_1} \omega & \text{for } k=1, \\ 0 & \text{otherwise.} \end{cases}$$

## A.3 Hochschild and cyclic (co)homology

The Hochschild cochain complex  $C^\bullet(A, A)$  of the unital algebra  $A$  with values in the  $A$ -bimodule  $M$  is defined as

$$C^k(A, M) = \text{Hom}(A^{\otimes k}, M).$$

The Hochschild coboundary operator  $d_H$  is given by

$$\begin{aligned} (d_H \Psi)(a_1, \dots, a_{n+1}) &= (-1)^{n+1} a_1 \Psi(a_2, \dots, a_n) \\ &\quad + \sum_{j=1}^n (-1)^{j+n+1} \Psi(a_1, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_{n+1}) \\ &\quad + \Psi(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

There is a Lie bracket  $[\cdot, \cdot]_G$  on  $C^{\bullet+1}(A, A)$ , called the Gerstenhaber bracket. It is defined as

$$[\Psi, \Phi]_G = \Psi \circ \Phi - (-1)^{(m-1)(n-1)} \Phi \circ \Psi,$$

where  $\Psi \in C^m(A, A)$ ,  $\Phi \in C^n(A, A)$  and

$$\begin{aligned} (\Psi \circ \Phi)(a_1, \dots, a_{n+m-1}) \\ = \sum_{j=1}^m (-1)^{(n-1)(j-1)} \Psi(a_1, \dots, a_{j-1}, \Phi(a_j, \dots, a_{j+n-1}), a_{j+n}, \dots, a_{n+m-1}). \end{aligned}$$

If we set

$$m(a_1, a_2) = a_1 \cdot a_2,$$

so that  $m \in C^1(A, A)$ , one can check that  $d_H(\cdot) = [m, \cdot]_G$ . Hence, by the Jacobi identity for  $[\cdot, \cdot]_G$ ,  $C^{\bullet+1}(A, A)$  is a dgla, and hence an  $L_\infty$ -algebra.

The normalized Hochschild chain complex  $C_\bullet(A, M)$  with values in the bimodule  $M$  is defined as

$$C_{-k}(A, M) = M \otimes \bar{A}^{\otimes k},$$

where  $\bar{A} = A/(1 \cdot \mathbb{C})$ .

**Remark A.2.** In the case of interest to us, that is, for  $M = A = C^\infty(M)$  being the locally convex algebra of smooth functions on a manifold  $M$ , one has to understand tensor products as projectively completed tensor products, see [2, Part II, Sections 5 and 6,



125 pp.]. Concretely, one obtains

$$C_{-k}(A, A) \cong C^\infty(M^{k+1})/N,$$

where  $N$  is the subspace of functions constant in one of the last  $k$  arguments. Alternatively, one can define  $C_{-k}(A, A)$  either as the  $\infty$ -jets or germs at the diagonal of functions in  $C^\infty(M^{k+1})$ , modulo jets or germs of functions constant in one of the last  $k$  arguments, see [11, Remark 3.1.1].  $\square$

In any case, the differential is defined by the formula

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{j=1}^{n-1} (-1)^j m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n \cdot m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The action (1) endows  $C_\bullet(A, A)$  with the structure of a differential graded module over  $C^{\bullet+1}(A, A)$ . On  $C_\bullet(A, A)$  there is another natural operation, namely the  $B$  of (3). One can check that  $B$  anticommutes with  $b$ , so that it makes sense to define the cyclic chain complex  $(CC_\bullet^W(A, A), b + uB)$  as in (2). Depending on the choice of the  $\mathbb{C}((u))$ -module  $W$  one obtains different cyclic homology theories:

- For  $W = \mathbb{C}$  with  $u$  acting as 0 one recovers the usual Hochschild chain complex.
- For  $W = \mathbb{C}((u))$  one obtains the periodic cyclic chain complex  $CC_\bullet^{\text{per}}(A, A)$ . In the case  $A = C^\infty(M)$ , it is quasi-isomorphic to the complex  $(\Omega^\bullet(M)((u)), d)$ , whose cohomology is  $H^\bullet(M)((u))$ .

Furthermore  $B$  (graded) commutes with the action of  $C^{\bullet+1}(A, A)$ , and hence the cyclic chain complex carries the structure of a differential graded  $C^{\bullet+1}(A, A)$ -module.

## References

- [1] Block, J. and E. Getzler. "Equivariant cyclic homology and equivariant differential forms." *Annales Scientifiques de l'Ecole Normale Supérieure* (4) 27, no. 4 (1994): 493–527.
- [2] Connes, A. "Non-commutative differential geometry." *Institut de Hautes Études Scientifiques. Publications Mathématiques* 62, no. 1 (1985): 41–144.
- [3] Dolgushev, V. "A formality theorem for Hochschild chains." *Advances in Mathematics* 200, no. 1 (2006): 51–101.

- [4] Dolgushev, V., D. Tamarkin, and B. Tsygan. "Formality of the homotopy calculus algebra of Hochschild (co)chains." (2008): preprint arXiv:0807.5117.
- [5] Getzler, E. and J. D. S. Jones. " $A_\infty$ -algebras and the cyclic bar complex." *Illinois Journal of Mathematics* 34 (1990): 256–83.
- [6] Hinich, V. "Tamarkin's proof of Kontsevich formality theorem." *Forum Math.* 15, (2003) 591–614.
- [7] Kontsevich, M. "Deformation quantization of Poisson manifolds." *Letters in Mathematical Physics* 66, no. 3 (2003): 157–216.
- [8] Shoikhet, B. "A proof of the Tsygan formality conjecture for chains." *Advances in Mathematics* 179, no. 1 (2003): 7–37.
- [9] Tamarkin, D. "Another proof of M. Kontsevich formality theorem, 1998." (1998): preprint arXiv:math/9803025.
- [10] Tamarkin, D. and B. Tsygan. "Noncommutative differential calculus, homotopy BV algebras and formality conjectures." *Methods of Functional Analysis and Topology* 6, no. 2 (2000), 85–100.
- [11] Tsygan, B. "Formality conjectures for chains." In *Differential topology, infinite-dimensional Lie algebras, and applications*, 261–74. American Mathematical Society Translation Series 2, 194. Providence, RI: American Mathematical Society, 1999.